IMPACT: International Journal of Research in Applied, Natural and Social Sciences (IMPACT: IJRANSS)
ISSN(P): 2347-4580; ISSN(E): 2321-8851
Vol. 4, Issue 8, Aug 2016, 59-72
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NATURAL DECOMPOSITION METHOD FOR SOLVING KLEIN GORDON EQUATIONS

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ABSTRACT

In this research paper, the Natural Decomposition Method (NDM) is implemented for solving the linear and nonlinear Klein Gordon equations.

The method which is based on the Natural Transform Method (NTM) and the Adomian Decomposition Method (ADM) is use to obtain exact solutions of three modelling problems from Mathematical Physics.

The results obtained are in agreement with existing solutions obtained by other methods and demonstrate the simplicity and efficiency of the NDM.

KEYWORDS: Adomian Decomposition Method, Klein Gordon Equations, Natural Transform, Sumudu Transform, Laplace Transform

1. INTRODUCTION

The Klein Gordon Equation is considered one of the most important Mathematical models in quantum field theory, with appearances in relativistic physics, nonlinear optics and plasma physics. It arises in physics in linear and nonlinear forms and it is useful in describing disperse wave phenomenon [1].

We consider the Klein Gordon Equation

$$u_{tt}(x,t)-u_{xx}(x,t)+au(x,t)+F(u(x,t))=h(x,t) \quad \textbf{(1.1)}$$

subject to the initial conditions

$$u(x,0)=f(x) \;\; ; \quad u_t(x,0)=g(x) \quad \textbf{(1.2)}$$

Where u is a function of x and t, a is a constant, $h(x,t)$ is a known analytic function and $F(u(x,t))$ is a nonlinear function of $u(x,t)$

There are many integral transform methods [3-7] existing in the literature to solve PDEs, ODEs and integral equations. Many numerical methods were developed recently for solving Klein Gordon Equations such as Reduced Differential Transform Method (RDTM) [2], Adomian Decomposition Method (ADM) [8,9]and Variational Iteration Method (VIM) [10,11].

In this paper, the following Klein Gordon Equations were solved:

First, consider the homogenous Klein Gordon Equation [10]

$$u_{tt} - u_{xx} - u = 0 \tag{1.3}$$

Subject to the initial conditions

$$u(x,0) = 1 + \sin x \ ; \ \ u_t(x,0) = 0 \tag{1.4}$$

Where $u = u(x,t)$ is a function of the variables x and t.

Secondly, the in homogenous nonlinear Klein Gordon Equation [2]

$$u_{tt} - u_{xx} + u^2 = -x\cos(t) + x^2\cos^2(t) \tag{1.5}$$

subject to the initial conditions

$$u(x,0) = x \ \ ; \ u_t(x,0) = 0 \tag{1.6}$$

and lastly, the nonlinear non-homogenous Klein Gordon equation [1]

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4 \tag{1.7}$$

Subject to the initial conditions

$$u(x,0) = u_t(x,0) = 0 \tag{1.8}$$

The structure of the paper is organized as follows: In section 2, basic idea of the Natural Transform Method is discussed, section 3 give definitions and properties of the N-Transform. In section 4, the methodology of the NDM is explained and in section 5, the NDM is applied to solve three test examples in order to show its simplicity and efficiency. Section 6 is the conclusion.

2. BASIC IDEA OF THE NATURAL TRANSFORM METHOD

Here we discuss some preliminaries about the nature of the Natural Transform Method (NTM).

Consider a function $f(t),\ t \in (-\infty, \infty)$, then the general Integral transform is defined as follows [7, 12]:

$$\Im[f(t)](s) = \int_{-\infty}^{\infty} k(s,t) f(t) dt \tag{2.1}$$

where $k(s,t)$ represent the kernel of the transform, s is the real (complex) number which is independent of t.

Note that when $k(s,t)$ is e^{-st}, $tJ_n(st)$ and $t^{s-1}(st)$, then Equation (2.1) gives, respectively, Laplace Transform, Hankel Transform and Mellin Transform.

Now, for $f(t),\ t \in (-\infty, \infty)$ consider the Integral transforms defined by

$$\Im[f(t)](u)=\int_{-\infty}^{\infty}k(t)f(ut)dt \tag{2.2}$$

and $$\Im[f(t)](s,u)=\int_{-\infty}^{\infty}k(s,t)f(ut)dt \tag{2.3}$$

Note that:

- when $k(t)=e^{-t}$, Equation (2.2) gives the Integral Sumudu transform, where parameter s is replaced by u. Moreover, for any value of n, the generalized Laplace and Sumudu transform are respectively defined by [7,12]:

$$\ell[f(t)]=F(s)=s^{n}\int_{0}^{\infty}e^{-s^{n+1}t}f(s^{n}t)dt \tag{2.4}$$

and $$S[f(t)]=G(u)=u^{n}\int_{0}^{\infty}e^{-u^{n}t}f(tu^{n+1})dt \tag{2.5}$$

- when $n=0$, Equation (2.4) and Equation (2.5) are the Laplace and Sumudu transform respectively.

3. DEFINITIONS AND PROPERTIES OF THE N-TRANSFORM

The natural transform of the function $f(t),\ t\in(-\infty,\infty)$ is defined by [7, 12]:

$$\mathrm{N}[f(t)]=R(s,u)=\int_{-\infty}^{\infty}e^{-st}f(ut)dt\ ;\quad s,u\in(-\infty,\infty) \tag{3.1}$$

where $\mathrm{N}[f(t)]$ is the natural transformation of the time function $f(t)$ and the variable s and u are the natural transform variables.

Note:

- Equation (3.1) can be written in the form [7,12]:

$$\mathrm{N}[f(t)]=\int_{-\infty}^{\infty}e^{-st}f(ut)dt\ ;\quad s,u\in(-\infty,\infty)$$

$$=\left[\int_{-\infty}^{0}e^{-st}f(ut)dt\ ;s,u\in(-\infty,0)\right]+\left[\int_{0}^{\infty}e^{-st}f(ut)dt\ ;s,u\in(0,\infty)\right]$$

$$=\mathrm{N}^{-}[f(t)]+\mathrm{N}^{+}[f(t)]=\mathrm{N}[f(t)\mathrm{H}(-t)]+\mathrm{N}[f(t)\mathrm{H}(t)]=R^{-}(s,u)+R^{+}(s,u)$$

where $\mathrm{H}(\cdot)$ is the Heaviside function.

- If the function $f(t)\mathrm{H}(t)$ is defined on the positive real axis, with $t \in (0,\infty)$ and in the set $A = \left\{ f(t) :\ni M, \tau_1, \tau_2 > 0, s.t \left| f(t) \right| < Me^{\frac{|t|}{\tau_j}}, \text{where } t \in (-1)^j \times [0,\infty),\ j = 1,2 \right\}$, then we define the Natural transform (N-Transform) as [3,4]:

$$\mathrm{N}[f(t)\mathrm{H}(t)] = \mathrm{N}^+[f(t)] = R^+(s,u) = \int_0^\infty e^{-st} f(ut)dt \ ; \quad s,u \in (0,\infty) \tag{3.2}$$

- If $u = 1$, Equation (3.2) can be reduced to the Sumudu transform. Now, we give some of the N-Transforms and the conversion to Sumudu and Laplace [7,12].

Table 1: Special N-Transforms and the Conversion to Sumudu and Laplace

$f(t)$	$\mathrm{N}[f(t)]$	$\mathrm{S}[f(t)]$	$\ell[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n = 1,2,\ldots$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\cos t$	$\frac{s}{s^2+u^2}$	$\frac{1}{1+u^2}$	$\frac{s}{1+s^2}$
$\sin t$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Some basic properties of the N-Transform are given as follows [7, 12]:

- If $R(s,u)$ is the natural transform and $F(s)$ is the Laplace transform of the function $f(t)$, then $\mathrm{N}^+[f(t)] = R(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)dt = \frac{1}{u}F\left(\frac{s}{u}\right).$

- If $R(s,u)$ is the natural transform and $G(u)$ is the Sumudu transform of the function $f(t)$, then $\mathrm{N}^+[f(t)] = R(s,u) = \frac{1}{s}\int_0^\infty e^{-t} f\left(\frac{ut}{s}\right)dt = \frac{1}{s}G\left(\frac{u}{s}\right).$

- If $\mathrm{N}^+[f(t)] = R(s,u)$, then $\mathrm{N}^+[f(at)] = \frac{1}{a}R(s,u).$

- If $\mathrm{N}^+[f(t)] = R(s,u)$, then $\mathrm{N}^+[f'(t)] = \frac{s}{u}R(s,u) - \frac{f(0)}{u}.$

- If $\mathrm{N}^{+}[f(t)] = R(s,u)$, then $\mathrm{N}^{+}[f''(t)] = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u}$.
- Linearity property [12]: If a and b are non-zero constants, and $f(t)$ and $g(t)$ are functions, then $\mathrm{N}^{+}[af(t) \pm bg(t)] = a\,\mathrm{N}^{+}[f(t)] \pm b\,\mathrm{N}^{+}[g(t)] = a\,\mathrm{F}^{+}(s,u) \pm b\,\mathrm{G}^{+}(s,u)$.

  Moreover, $\mathrm{F}^{+}(s,u)$ and $\mathrm{G}^{+}(s,u)$ are the N-transforms of $f(t)$ and $g(t)$ respectively.

4. THE NATURAL DECOMPOSITION METHOD

The applicability of the natural decomposition method to Klein Gordon Equation is illustrated as follows:

Consider the following Klein Gordon Equations (1.1)

$$L_t u(x,t) - L_x u(x,t) + au(x,t) + F(u(x,t)) = h(x,t) \tag{4.1}$$

Subject to initial conditions

$$u(x,0) = f(x)\ ;\ u_t(x,0) = g(x) \tag{4.2}$$

where $L_t = \frac{\partial^2}{\partial t^2}$, $L_x = \frac{\partial^2}{\partial x^2}$, $Fu(x,t)$ is a nonlinear function of $u(x,t)$ and $h(x,t)$ is a known analytic function.

By taking the N-transform of Equation (4.1), we have

$$\mathrm{N}^{+}[L_t u(x,t)] - \mathrm{N}^{+}[L_x u(x,t)] + a\,\mathrm{N}^{+}[u(x,t)] + \mathrm{N}^{+}[F(u(x,t))] = \mathrm{N}^{+}[h(x,t)]$$

Using the properties in Table 1 and the basic properties of the N-transforms, we get

$$\frac{s^2}{u^2}R(x,s,u) - \frac{su(x,0)}{u^2} - \frac{u'(x,0)}{u} - \mathrm{N}^{+}[L_x[u(x,t)]] + a\,\mathrm{N}^{+}[u(x,t)] + \mathrm{N}^{+}[f(u(x,t))] = \mathrm{N}^{+}[h(x,t)] \tag{4.3}$$

substituting Equation (4.2) into Equation (4.3) to get

$$\frac{s^2}{u^2}R(x,s,u) - \frac{sf(x)}{u^2} - \frac{g(x)}{u} - \mathrm{N}^{+}[L_x[u(x,t)]] + a\,\mathrm{N}^{+}[u(x,t)] + \mathrm{N}^{+}[f(u(x,t))] = \mathrm{N}^{+}[h(x,t)] \tag{4.4}$$

$$\frac{s^2}{u^2}R(x,s,u) = \frac{sf(x)}{u^2} + \frac{g(x)}{u} + \mathrm{N}^{+}[L_x[u(x,t)]] + \mathrm{N}^{+}[h(x,t)] - a\,\mathrm{N}^{+}[u(x,t)] - \mathrm{N}^{+}[f(u(x,t))] \tag{4.5}$$

$$R(x,s,u) = \frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}\mathrm{N}^{+}[L_x[u(x,t)]] + \frac{u^2}{s^2}\mathrm{N}^{+}[h(x,t)] - \frac{au^2}{s^2}\mathrm{N}^{+}[u(x,t)] - \frac{u^2}{s^2}\mathrm{N}^{+}[f(u(x,t))] \tag{4.6}$$

Taking the inverse natural transform of Equation (4.6), we have

$$\mathrm{N}^{-1}R(x,s,u)=\mathrm{N}^{-1}\left[\frac{f(x)}{s}+\frac{ug(x)}{s^2}+\frac{u^2}{s^2}\mathrm{N}^{+}[h(x,t)]\right]+\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^{+}[L_x[u(x,t)]]\right]$$
$$-\mathrm{N}^{-1}\left[a\frac{u^2}{s^2}\mathrm{N}^{+}[u(x,t)]+\frac{u^2}{s^2}\mathrm{N}^{+}[f(u(x,t))]\right] \quad \textbf{(4.7)}$$

From Equation (4.7), we have

$$u(x,t)=J(x,t)+\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^{+}[L_x(u(x,t))]\right]-\mathrm{N}^{-1}[ku(x,t)] \quad \textbf{(4.8)}$$

Where $ku(x,t)=a\frac{u^2}{s^2}\mathrm{N}^{+}(u(x,t))+\frac{u^2}{s^2}\mathrm{N}^{+}[f(u(x,t))]$ and $J(x,t)$ represents the term arising from the known analytical function and the given initial conditions.

Now to deal with the nonlinear term, we represent the solution in an infinite series form.

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \quad \textbf{(4.9)}$$

also, the nonlinear term

$ku(x,t)$ can be written as $ku(x,t)=\sum_{n=0}^{\infty}A_n$ **(4.10)**

Where the A_n's are the polynomials of $u_0, u_1, \ldots, u_n$ and can be calculated by the formula [4]

$$A_n=\frac{1}{n!}\frac{d^n}{dx^n}\left[f\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right],\ n=0,1,2,\ldots \quad \textbf{(4.11)}$$

Substituting Equation (4.9) and Equation (4.10) into Equation (4.8) to get

$$\sum_{n=0}^{\infty}u_n(x,t)=J(x,t)+\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^{+}\left[L_x\sum_{n=0}^{\infty}u_n(x,t)\right]-\sum_{n=0}^{\infty}A_n\right] \quad \textbf{(4.12)}$$

by comparing both sides of Equation (4.2) we conclude that

$$u_0(x,t)=J(x,t)$$
$$u_1(x,t)=\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^{+}(L_x u_0(x,t)-A_0)\right]$$
$$u_2(x,t)=\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^{+}(L_x u_1(x,t))-A_1\right]$$

Continuing in this manner, we get the general recursive relation given by:

$$u_{n+1}(x,t) = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^{+}\left(L_x u_n(x,t)\right) - A_n\right], \; n \geq 1 \tag{4.13}$$

Hence, from the general recursive relation in Equation (4.13), we can easily compute the remaining components of $u(x,t)$ as $u_1(x,t), u_2(x,t), \ldots,$ where $u_0(x,t)$ is always the given initial conditions. Finally the exact solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

.

5. APPLICATIONS

Here, we employ the NDM to three numerical examples and then compare our solutions to existing exact solutions.

Example 5.1: Consider the homogenous Klein Gordon Equation

$$u_{tt} - u_{xx} - u = 0 \tag{5.1}$$

Subject to the initial conditions

$$u(x,0) = 1 + \sin x \;,\; u_t(x,0) = 0 \tag{5.2}$$

We first take the N-Transform of Equation (5.1), to obtain

$$\mathrm{N}^{+}[u_{tt}] - \mathrm{N}^{+}[u_{xx}] - \mathrm{N}^{+}[u] = \mathrm{N}^{+}[0]$$

Using the properties in Table 1 and properties of the N-Transform, we have

$$\frac{s^2}{u^2}R(x,s,u) - \frac{su(x,0)}{u^2} - \frac{u'(x,0)}{u} - \mathrm{N}^{+}[u_{xx}] - \mathrm{N}^{+}[u] = 0 \tag{5.3}$$

Substituting Equation (5.2) into Equation (5.3), we have

$$R(x,s,u) = \frac{(1+\sin x)}{s} + \frac{u^2}{s^2}\mathrm{N}^{+}[u_{xx} + u] \tag{5.4}$$

Now, taking the inverse N-Transform of Equation (5.4), we have

$$N^{-1}[R(x,s,u)] = N^{-1}\left[\frac{1+\sin x}{s}\right] + N^{-1}\left[\frac{u^2}{s^2}N^{+}[u_{xx}+u]\right] \quad \textbf{(5.5)}$$

From Table 1, Equation (5.5) becomes

$$u(x,t) = (1+\sin x) + N^{-1}\left[\frac{u^2}{s^2}N^{+}[u_{xx}+u]\right] \quad \textbf{(5.6)}$$

From Equation (5.6), we can write

$$u(x,t) = 1+\sin x + N^{-1}\left[\frac{u^2}{s^2}N^{+}\left[\sum_{n=0}^{\infty}B_n + \sum_{n=0}^{\infty}A_n\right]\right] \quad \textbf{(5.7)}$$

Now from Equation (5.7), we can conclude that

$$u_0(x,t) = 1+\sin x, \quad u_1(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^{+}[B_0 + A_0]\right]$$

We continue in this manner to get the general recursive relation

$$u_{n+1}(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^{+}[B_n + A_n]\right], \quad n \geq 1. \quad \textbf{(5.8)}$$

$$u_1(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^{+}[B_0 + A_0]\right] = N^{-1}\left[\frac{u^2}{s^2}N^{+}[u_{0xx} + u_0]\right]$$

Note that

$$= N^{-1}\left[\frac{u^2}{s^2}N^{+}[1]\right] = N^{-1}\left[\frac{u^2}{s^2}\left(\frac{1}{s}\right)\right] = N^{-1}\left[\frac{u^2}{s^3}\right] = \frac{1}{2}t^2.$$

$$u_{n+1}(x,t) = 0, \ \forall n \geq 1. \text{Hence } u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots = 1 + \sin x + \frac{1}{2}t^2.$$

and that

$$u(x,t) = \sin x + \cosh t.$$

Hence the exact solution is [1].

Example 5.2: Consider the inhomogenous nonlinear Klein Gordon Equation

$$u_{tt} - u_{xx} + u^2 = -x\cos t + x^2\cos^2 t \quad \textbf{(5.9)}$$

Subject to the initial conditions

$$u(x,0) = x \ , \ u_t(x,0) = 0 \tag{5.10}$$

We first take the N-Transform of Equation (5.1), to obtain

$$\mathrm{N}^+[u_{tt}] - \mathrm{N}^+[u_{xx}] + \mathrm{N}^+[u^2] = \mathrm{N}^+[-x\cos t + x^2\cos^2 t]$$

Using the properties in Table 1 and properties of the N-Transform, we have

$$\frac{s^2}{u^2}R(x,s,u) - \frac{su(x,0)}{u^2} - \frac{u'(x,0)}{u} - \mathrm{N}^+[u_{xx}] + \mathrm{N}^+[u^2] = \mathrm{N}^+[-x\cos t] + \mathrm{N}^+[\mathrm{x}^2\cos^2 t] \tag{5.11}$$

Substituting Equation (5.10) into Equation (5.11), we have

$$R(x,s,u) = \frac{x}{s} - \frac{u^2 x}{s^2}\left(\frac{s}{s^2+u^2}\right) + \frac{u^2x^2}{s^2}\left(\frac{s}{s^2+u^2}\right)^2 + \frac{u^2}{s^2}\mathrm{N}^+[u_{xx} - u^2] \tag{5.12}$$

Now, taking the inverse N-Transform of Equation (5.12), we have

$$\mathrm{N}^{-1}[R(x,s,u)] = \mathrm{N}^{-1}\left[\frac{x}{s}\right] - \mathrm{N}^{-1}\left[\frac{u^2x}{s(s^2+u^2)}\right] + \mathrm{N}^{-1}\left[\frac{u^2x^2}{(s^2+u^2)^2}\right] + \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[u_{xx} - u^2]\right] \tag{5.13}$$

From Table 1, Equation (5.13) becomes

$$u(x,t) = x\cos t + x^2\sin^2 t + \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[u_{xx} - u^2]\right] \tag{5.14}$$

From Equation (5.14), we can write

$$u(x,t) = x\cos t + x^2\sin^2 t + \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left[\sum_{n=0}^{\infty}B_n - \sum_{n=0}^{\infty}A_n\right]\right] \tag{5.15}$$

Here, A_n is the Adomian polynomial which represent the nonlinear terms. So, we compute few components of A_n and some values of B_n.

$$\begin{aligned} A_0 &= u_0{}^2 & B_0 &= u_{0xx} \\ A_1 &= 2u_0u_1 & B_1 &= u_{1xx} \\ A_2 &= 2u_0u_2 + u_1{}^2 & B_2 &= u_{2xx} \end{aligned}$$

Now from Equation (5.15), we can conclude that

$$u_0(x,t) = x\cos t + x^2 \sin^2 t\,, \quad u_1(x,t) = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[B_0 - A_0]\right]$$

We continue in this manner to get the general recursive relation

$$u_{n+1}(x,t) = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[B_n - A_n]\right], \quad n \geq 1. \tag{5.16}$$

Note that we can calculate

$$u_1(x,t) = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[B_0 - A_0]\right] = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left[u_{0xx} - u_0{}^2\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left(2\sin^2 t - x^2\cos^2 t - 2x^3\cos t\sin^2 t + x^4\sin^4 t\right)\right] = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left[-x^2\cos^2 t\right]\right] + \ldots$$

$$= -x^2\,\mathrm{N}^{-1}\left[\left(\frac{u}{s^2+u^2}\right)^2\right] + \ldots = -x^2\sin^2 t + \ldots$$

Hence by cancelling the noise term that appears between $u_0(x,t)$ and $u_1(x,t)$ one can see that the non-cancelled term of $u_0(x,t)$ still satisfies the given differential equation, which lead to an exact solution of the form

$$u(x,t) = x\cos t.$$

This is in agreement with the result obtained by RDTM [2].

Example 5.3: Consider the nonlinear, nonhomogenous Klein Gordon Equation

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4 t^4 \tag{5.17}$$

Subject to the initial conditions

$$u(x,0) = u_t(x,0) = 0 \tag{5.18}$$

We first take the N-Transform of Equation (5.17), to obtain

$$\mathrm{N}^+[u_{tt}] - \mathrm{N}^+[u_{xx}] + \mathrm{N}^+\left[u^2\right] = \mathrm{N}^+\left[2x^2\right] - \mathrm{N}^+\left[2t^2\right] + \mathrm{N}^+\left[x^4 t^4\right]$$

Using the properties in Table 1 and properties of the N-Transform, we have

$$\frac{s^2}{u^2}R(x,s,u)-\frac{su(x,0)}{u^2}-\frac{u'(x,0)}{u}-\mathrm{N}^+[u_{xx}]+\mathrm{N}^+[u^2]=\mathrm{N}^+[2x^2]-\mathrm{N}^+[2t^2]+\mathrm{N}^+[x^4t^4] \tag{5.19}$$

Substituting Equation (5.18) into Equation (5.19), we have

$$R(x,s,u)=2x^2\frac{u^2}{s^3}-4\frac{u^4}{s^5}+24x^4\frac{u^6}{s^7}+\frac{u^2}{s^2}\mathrm{N}^+[u_{xx}-u^2] \tag{5.20}$$

Now taking the inverse N-Transform of Equation (5.20), we have

$$\mathrm{N}^{-1}[R(x,s,u)]=2x^2\,\mathrm{N}^{-1}\left[\frac{u^2}{s^3}\right]-4\,\mathrm{N}^{-1}\left[\frac{u^4}{s^5}\right]+24x^4\,\mathrm{N}^{-1}\left[\frac{u^6}{s^7}\right]+\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[u_{xx}-u^2]\right] \tag{5.21}$$

From Table 1, Equation (5.21) becomes

$$u(x,t)=x^2t^2-\frac{1}{6}t^4+\frac{1}{30}x^4t^6+\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[u_{xx}-u^2]\right] \tag{5.22}$$

From Equation (5.22), we can write

$$u(x,t)=x^2t^2-\frac{1}{6}t^4+\frac{1}{30}x^4t^6+\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left[\sum_{n=0}^{\infty}B_n-\sum_{n=0}^{\infty}A_n\right]\right] \tag{5.23}$$

Here, A_n is the Adomian polynomial which represent the nonlinear terms. So, we compute few components of A_n and some values of B_n.

$$\begin{aligned} A_0 &= u_0^{\;2} & B_0 &= u_{0xx} \\ A_1 &= 2u_0u_1 & B_1 &= u_{1xx} \\ A_2 &= 2u_0u_2+u_1^{\;2} & B_2 &= u_{2xx} \end{aligned}$$

Now from Equation (5.23), we can conclude that

$$u_0(x,t)=x^2t^2-\frac{1}{6}t^4+\frac{1}{30}x^4t^6\,,\quad u_1(x,t)=\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[B_0-A_0]\right]$$

We continue in this manner to get the general recursive relation

$$u_{n+1}(x,t)=\mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[B_n-A_n]\right],\quad n\geq 1. \tag{5.24}$$

Note that we can calculate

$$u_1(x,t) = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+[B_0 - A_0]\right] = \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left[u_{0xx} - u_0{}^2\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left[2t^2 + \frac{2}{5}x^2t^6 - x^4t^4 + \frac{2}{3}x^2t^6 - \frac{1}{15}x^6t^8 + \frac{1}{90}x^4t^{10} - \frac{1}{36}t^8 - \frac{1}{900}x^8t^{12}\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u^2}{s^2}\mathrm{N}^+\left(2t^2 - x^4t^4\right)\right] + \ldots \quad = \mathrm{N}^{-1}\left[4\frac{u^4}{s^5} - 24x^4\frac{u^6}{s^7}\right] + \ldots \quad = \frac{1}{6}t^4 - \frac{1}{30}x^4t^6 + \ldots$$

Hence by cancelling the noise term that appears between $u_0(x,t)$ and $u_1(x,t)$ one can see that the non-cancelled term of $u_0(x,t)$ still satisfies the given differential equation, which lead to an exact solution of the form

$$u(x,t) = x^2t^2.$$

This is in agreement with the result obtained by MDM [1].

6. CONCLUSIONS

In this research paper, the Natural Decomposition Method (NDM) was applied to solve three nonlinear Klein Gordon Equations. Exact solutions of the three applications were obtained. The method demonstrates significant improvement over existing techniques.

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